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Stochastic quantization of order two parafermi fields

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Abstract. The method of stochastic quantization of Parisi–Wu is extended to include spinor fields obeying the generalized statistics of order two consistent with the weak locality requirement. Appropriate Langevin and Fokker–Planck equations are constructed using paragrassmann variables, which give rise to two fields with different masses in the equilibrium limit, in agreement with the results of the canonical quantization procedure. The connection between the stochastic quantization method and conventional Euclidean field theory is established through Klein transformations.

1 Introduction

It is well known in quantum mechanics that in four dimensions, if the interchange of two particles does not lead to a new state, then the particles must satisfy either Fermi–Dirac or Bose–Einstein statistics depending upon the symmetry character of the wave function. On the other hand, if such an interchange does lead to a new state, then the particles would satisfy "parastatistics" – statistics more general than Fermi or Bose statistics, and the Schrödinger N-body wave functions for such systems, although quantum-mechanically acceptable, will be neither symmetric nor antisymmetric [1]. Applying these ideas, Greenberg [2] suggested that quarks could be parafermions of order three, the order of the statistics providing an alternative to explicit colour symmetry.

Quantization of systems obeying parastatistics using canonical and path integral methods is an old subject and has been discussed in detail in [3] and references therein.

Parisi and Wu introduced their method of stochastic quantization [4] as an alternative to the path integral formalism. Their method has been applied so far to quantize scalar fields [4], ordinary Fermi fields [5], gauge fields [4,6], gravity [7], and parafermi fields which are of odd orders satisfying the locality condition of the weak form and those of any order satisfying the locality condition of the strong form [8]. The pathological case of order two parafermi fields was not discussed in [8].

In the Parisi–Wu scheme [9,10], one introduces a fifth dimension called the fictitious time, in addition to the usual space-time dimensions in the corresponding Euclidean field theory and postulates a non-equilibrium stochastic Langevin dynamics for the system. In the limit of the fictitious time going to infinity, corresponding to the equilibrium limit of the stochastic system, the usual Euclidean field theory is reproduced. This is effected by first proving the equivalence of the Langevin equation with the corresponding Fokker–Planck equation.

In this paper it is shown how to extend the Parisi–Wu method to include parafermi fields of order two consistent with the weak locality requirement in four space-time dimensions. In order to achieve this, we modify both the Langevin equation as well as the corresponding Fokker–Planck equation in an appropriate manner and show that one can reproduce with these modified equations, the same result for the propagator as is obtained with the canonical method.

It is known that the normal Fermi–Dirac bilinear commutation rules for spinor field operators $\psi(x)$ and $\bar{\psi}(x)$ satisfy the following trilinear commutation rules obtained from the Heisenberg equations of motion:

$$\begin{aligned}[][\psi(x),[\psi^\dagger(y),\psi^\dagger(z)]] &= 2\delta^4(x-y)\psi^\dagger(z) - 2\delta^4(x-z)\psi^\dagger(y),\\ [\psi(x),[\psi^\dagger(y),\psi(z)]] &= 2\delta^4(x-y)\psi(z),\\ [\psi(x),[\psi(y),\psi(z)]] &= 0,\end{aligned} \tag{1}$$

where x, y and z denote space-time variables. In terms of the creation and the annihilation operators, these trilinear relations assume the form

$$\begin{aligned}[][b_i,[b^\dagger{}_j,b^\dagger{}_k]] &= 2\delta_{ij}b^\dagger{}_k - 2\delta_{ik}b^\dagger{}_j,\\ [b_i,[b^\dagger{}_j,b_k]] &= 2\delta_{ij}b_k,\\ [b_i,[b_j,b_k]] &= 0,\end{aligned} \tag{2}$$

However, there are a host of other possibilities [1] which also satisfy these trilinear relations. These may be represented by the operators ${b_i}^\alpha$ which satisfy commutation relations of the anomalous kind:

$$\begin{aligned}[][{b_i}^\alpha,{b_j}^\alpha]_+ &= [{b_i}^{\alpha\dagger},{b_j}^{\alpha\dagger}]_+ = 0,\\ [{b_i}^\alpha,{b_j}^{\alpha\dagger}]_+ &= \delta_{ij},\end{aligned}$$

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$$[b_i{}^\alpha, b_j{}^{\beta\dagger}]_- = [b_i{}^\alpha, b_j{}^\beta]_- = 0 \quad (\alpha \neq \beta). \tag{3}$$

The $b_i{}^\alpha$ are called the Green components of b_i of order p:

$$b_i = \sum_{\alpha=1}^{p} b_i{}^\alpha, \tag{4}$$

where the indices α, β are known as Green indices. Fields whose creation and annihilation operators obey the relations (3) are called parafermi fields. Equations (3) may be concisely written as

$$\begin{aligned}[b_i{}^\alpha, b_j{}^{\beta\dagger}]_{-\nu_{\alpha\beta}} &= \delta_{ij}\delta_{\alpha\beta}, \\ [b_i{}^\alpha, b_j{}^\beta]_{-\nu_{\alpha\beta}} &= 0,\end{aligned} \tag{5}$$

where

$$\nu_{\alpha\beta} = \begin{cases} -1 \text{ if } \alpha = \beta, \\ +1 \text{ if } \alpha \neq \beta. \end{cases}$$

It was shown in [8] that parafermi fields of order greater than two can be stochastically quantized starting from the following Langevin equations:

$$\begin{aligned}\frac{\partial \psi(x,t)}{\partial t} &= -\frac{\delta S}{\delta \bar{\psi}(x,t)} + \eta(x,t), \\ \frac{\partial \bar{\psi}(x,t)}{\partial t} &= -\frac{\delta S}{\delta \psi(x,t)} + \bar{\eta}(x,t),\end{aligned} \tag{6}$$

where t denotes the fictitious time, x includes all the space-time variables, $\psi(x,t)$ and $\bar{\psi}(x,t)$ are independent paragrassmann fields and $\eta(x,t)$ and $\bar{\eta}(x,t)$ are independent paragrassmann random noise sources. The action was taken to be bilinear in the fields:

$$S[\psi, \bar{\psi}] = \int \mathrm{d}x \frac{1}{2}[\bar{\psi}, K\psi]_-, \tag{7}$$

where K may in general contain derivative operators, scalar fields, γ-matrices, and the fields ψ and $\bar{\psi}$ neither commute nor anticommute, but constitute p Green components

$$\psi(x,t) = \sum_{\alpha=1}^{p} \psi^\alpha(x,t), \quad \bar{\psi}(x,t) = \sum_{\alpha=1}^{p} \bar{\psi}^\alpha(x,t), \tag{8}$$

satisfying the anomalous commutation relations

$$\begin{aligned}&[\psi^\alpha(x,t), \psi^\alpha(x',t')]_+ = [\psi^\alpha(x,t), \bar{\psi}^\alpha(x',t')]_+ \\ &\quad = [\bar{\psi}^\alpha(x,t), \bar{\psi}^\alpha(x',t')]_+ = 0, \\ &[\psi^\alpha(x,t), \psi^\beta(x',t')]_- = [\psi^\alpha(x,t), \bar{\psi}^\beta(x',t')]_- \\ &\quad = [\bar{\psi}^\alpha(x,t), \bar{\psi}^\beta(x',t')]_- = 0, \quad \alpha \neq \beta.\end{aligned} \tag{9}$$

In the following sections we construct the Langevin equations for parafermi fields of order two satisfying the weak locality requirement and prove its equivalence to its corresponding Fokker–Planck equation. Finally, we show the equivalence of the equations we have constructed to the standard Euclidean field theory in the equilibrium limit, through Klein transformations.

2 Quantization of parafermi fields of order two

Unlike the fermion case, any parafermion action cannot be cast in the form given in (7). The general form of the action for a parafield of any order is dictated by the locality condition [3].

The possible forms observables can take in a quantum field theory are severely restricted by the requirement that the theory be local. For an observable $F(V)$ (given as a functional of the field operator $\hat{\psi}(x)$, where $x \in V$) defined in the spatial region V, and another observable $F'(V')$ defined similarly for a spatial region V', the locality condition of the weak form dictates that the measurements of $F(V)$ and $F'(V')$ can be performed independently at the respective spatial regions V and V':

$$[F(V), F'(V')] = 0, \quad \text{for } V \sim V', \tag{10}$$

where $V \sim V'$ denotes the fact that V and V' are spatially far apart.

The strong locality condition on the other hand implies a relation such as

$$[\psi(y), F(V)] = 0, \quad \text{for } y \sim V, \tag{11}$$

according to which the measurement of $F(V)$ can be made independently of whether any particles exist at other points y which are spatially distant from V.

An action having the form (7) satisfies the strong locality condition for a parafermi field of any order and the weak locality condition for a parafermi field of odd order [3].

It was shown by Ohnuki and Kamefuchi [3] that for parafermi fields of order $p = 2$, the most general form of the action in four dimensions consistent with the weak locality requirement is

$$\begin{aligned}S_{\mathrm{E}}^{(\pm)} = \frac{1}{2}\int \mathrm{d}^4x \{ &-[\bar{\psi}(x), (\mathrm{i}\not{\nabla} - m)\psi]_- \\ &\pm \kappa[\bar{\psi}(x), \psi(x)]_+ \}.\end{aligned} \tag{12}$$

They showed that the state-vector space divides up into even and odd sectors defined with respect to the parity of the eigenvalues of the number operator N:

$$N \equiv \frac{1}{2}\int \mathrm{d}^3x [\psi^\dagger(x), \psi(x)], \tag{13}$$

satisfying $N|0\rangle = 0$, $|0\rangle$ denoting the vacuum state, and that these sectors are completely separated from each other. $S^{(+)}$ and $S^{(-)}$ in (12) denote the Euclidean action in the even and the odd sector respectively and κ is a non-vanishing real parameter. They also showed that a parafermi field of order two was equivalent to two ordinary Fermi fields of masses $(m \pm \kappa)$ and they used their results to describe the electron and the muon by a charged parafermi field of order two, and the neutrino pair ν_e and ν_μ by a neutral parafermi field of order two.

Because of the presence of the anticommutator term in the action, the Langevin equations (6) are not applicable for this order.

We postulate the following Langevin equations for the Green components of the parafermi fields of order two satisfying the weak locality condition:

$$\begin{aligned}\frac{\partial\psi^\alpha(x,t)}{\partial t} &= -\sum_\beta \theta_{\alpha\beta}\frac{\delta S_{\mathrm{E}}[\psi,\bar\psi]}{\delta\bar\psi^\beta(x,t)} + \eta^\alpha(x,t),\\ \frac{\partial\bar\psi^\alpha(x,t)}{\partial t} &= -\sum_\beta \nu_{\alpha\beta}\frac{\delta S_{\mathrm{E}}[\psi,\bar\psi]}{\delta\psi^\beta(x,t)} + \bar\eta^\alpha(x,t),\end{aligned}\tag{14}$$

where

$$\nu_{\alpha\beta} = \begin{cases} -1 \text{ if } \alpha=\beta,\\ +1 \text{ if } \alpha\neq\beta.\end{cases}$$

and

$$\theta_{\alpha\beta} = \nu_{\alpha\beta} + 2\delta_{\alpha\beta}.\tag{15}$$

In the summation in the first term on the right-hand side of (14), β spans all values, covering both $\alpha=\beta$ as well as $\alpha\neq\beta$. The fields $\psi^\alpha(x,t)$ and $\bar\psi^\alpha(x,t)$ are paragrassmannians satisfying (9). $\eta^\alpha(x,t)$ and $\bar\eta^\alpha(x,t)$ are paragrassmann gaussian random noise sources with the correlation properties

$$\begin{aligned}\langle\eta^\alpha(x,t)\rangle &= \langle\bar\eta^\alpha(x,t)\rangle = 0,\\ \langle\eta^\alpha(x,t)\bar\eta^\alpha(y,t')\rangle &= -\langle\bar\eta^\alpha(y,t')\eta^\alpha(x,t)\rangle\\ &= 2\delta(x-y)\delta(t-t'),\\ \left\langle\eta^\alpha(x,t)\bar\eta^\beta(y,t')\right\rangle &= \left\langle\bar\eta^\beta(y,t')\eta^\alpha(x,t)\right\rangle\\ &= 0,\quad \alpha\neq\beta,\\ \langle\eta^\alpha(x,t)\eta^\alpha(y,t')\rangle &= \langle\bar\eta^\alpha(x,t)\bar\eta^\alpha(y,t')\rangle\\ &= -\langle\eta^\alpha(y,t')\eta^\alpha(x,t)\rangle = 0,\\ \left\langle\eta^\alpha(x,t)\eta^\beta(y,t')\right\rangle &= \left\langle\bar\eta^\alpha(x,t)\bar\eta^\beta(y,t')\right\rangle\\ &= \left\langle\eta^\beta(y,t')\eta^\alpha(x,t)\right\rangle\\ &= 0,\quad \alpha\neq\beta, \text{etc.}\end{aligned}\tag{16}$$

It is important to observe that in (14), while the left-hand sides are time derivatives of the fields with respect to the Green component α, the right-hand sides contain functional derivatives of the Euclidean action with respect to the Green index $\beta\neq\alpha$, *as well as* with respect to $\beta=\alpha$. Thus, the time variation for the Green component $\alpha=1$ receives contributions from *both* $\alpha=1$, *and* from $\beta=2$.

In the Langevin approach, the quantum correlation functions are obtained by taking the $t\to\infty$ limit of the stochastic average of the fields

$$\left\langle\psi(x_1,t)\psi(x_2,t)\ldots\bar\psi(x_n,t)\right\rangle_{\eta\bar\eta},\tag{17}$$

where the angular bracket denotes the stochastic average of a function $f(\eta,\bar\eta)$ with respect to the random variables η and $\bar\eta$:

$$\langle f(\eta,\bar\eta)\rangle_{\eta,\bar\eta} = \frac{\int \mathrm{d}\bar\eta\mathrm{d}\eta f(\eta,\bar\eta)\exp\left\{-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}t\bar\eta(x,t)\eta(x,t)\right\}}{\int \mathrm{d}\bar\eta\mathrm{d}\eta\exp\left\{-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}t\bar\eta(x,t)\eta(x,t)\right\}}.\tag{18}$$

In Euclidean field theory the Green functions are regarded as moments or averages weighted with a probability distribution $\mathrm{e}^{-S_{\mathrm{E}}[\bar\psi,\psi]}$. In the Parisi–Wu formalism, the non-equilibrium stochastic process described by a probability distribution $P[\psi,\bar\psi,t]$ and the Langevin dynamics described above evolves in this fictitious (or fifth) time through the Fokker–Planck equation:

$$\frac{\partial P[\psi,\bar\psi,t]}{\partial t} = -H_{\mathrm{FP}}P[\psi,\bar\psi,t],\tag{19}$$

where H_{FP} is the Fokker–Planck hamiltonian. In order to prove that $P[\psi,\bar\psi,t]$ relaxes to the quantum distribution $\mathrm{e}^{-S_{\mathrm{E}}[\bar\psi,\psi]}$ in the equilibrium limit $t\to\infty$, one must show that H_{FP} is at least positive semi-definite.
We shall show that the Fokker–Planck equation which is equivalent to the Langevin equation constructed in (14) is given by

$$\begin{aligned}&\frac{\partial P[\psi,\bar\psi,t]}{\partial t}\\ &= \sum_\alpha\int \mathrm{d}^4x\sum_\beta\left\{\theta_{\alpha\beta}\frac{\delta}{\delta\bar\psi^\alpha}\left(\frac{\delta S[\bar\psi,\psi]}{\delta\psi^\beta} + \delta_{\alpha\beta}\frac{\delta}{\delta\psi^\beta}\right)\right.\\ &\quad\left.+\nu_{\alpha\beta}\frac{\delta}{\delta\psi^\beta}\left(\frac{\delta S[\bar\psi,\psi]}{\delta\bar\psi^\alpha} + \delta_{\alpha\beta}\frac{\delta}{\delta\bar\psi^\alpha}\right)\right\}P[\psi,\bar\psi,t],\end{aligned}\tag{20}$$

where the sum over α outside the integral on the right-hand side is performed at the end and gives the Fokker–Planck equation for the full parafermi field from its Green components. The equivalence with the Langevin equation (14) is proved in Sect. 4 by looking at the stochastic average of the variation in the fictitious time of an arbitrary functional F of ψ and $\bar\psi$. Applying the Langevin equations (14) one can write

$$\begin{aligned}\left\langle\frac{\partial F[\psi,\bar\psi]}{\partial t}\right\rangle_{\eta\bar\eta} &= \sum_\alpha\int \mathrm{d}^4x\left[\left\langle\frac{\partial\psi^\alpha}{\partial t}\frac{\delta F[\psi,\bar\psi]}{\delta\psi^\alpha}\right\rangle_{\eta\bar\eta}\right.\\ &\quad\left.+\left\langle\frac{\partial\bar\psi^\alpha}{\partial t}\frac{\delta F[\psi,\bar\psi]}{\delta\bar\psi^\alpha}\right\rangle_{\eta\bar\eta}\right]\\ &= \sum_\alpha\int \mathrm{d}^4x\left[-\sum_\beta\theta_{\alpha\beta}\left\langle\frac{\delta S_{\mathrm{E}}[\psi,\bar\psi]}{\delta\bar\psi^\beta(x,t)}\frac{\delta F[\psi,\bar\psi]}{\delta\psi^\alpha}\right\rangle_{\eta\bar\eta}\right.\\ &\quad-\sum_\beta\left\langle\nu_{\alpha\beta}\frac{\delta S_{\mathrm{E}}[\psi,\bar\psi]}{\delta\psi^\beta(x,t)}\frac{\delta F[\psi,\bar\psi]}{\delta\bar\psi^\alpha}\right\rangle_{\eta\bar\eta}\\ &\quad+\left\langle\eta^\alpha(x,t)\frac{\delta F[\psi,\bar\psi]}{\delta\psi^\alpha}\right\rangle_{\eta\bar\eta}\\ &\quad\left.+\left\langle\bar\eta^\alpha(x,t)\frac{\delta F[\psi,\bar\psi]}{\delta\bar\psi^\alpha}\right\rangle_{\eta\bar\eta}\right].\end{aligned}\tag{21}$$

In order to evaluate these averages, we prove the following theorem.

3 Novikov's theorem for paragrassmann variables

The proof depends upon a probability distribution over the Green components $\bar{\xi}_i^\alpha$ and ξ_i^β of independent paragrassmann variables $\bar{\xi}_i$ and ξ_i

$$P(\xi,\bar{\xi},t) = \exp\left(-\sum_{i,j}\sum_{\alpha,\beta}\bar{\xi}_i^\alpha A_{ij}\xi_j^\beta\right). \tag{22}$$

We define the average of a function F of $\bar{\xi}_i$ and ξ_i by

$$\langle F\rangle = \int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i F\mathrm{e}^{-\sum_{i,j}\bar{\xi}_i A_{ij}\xi_j}, \tag{23}$$

where

$$\left\langle \xi_i\bar{\xi}_k\right\rangle = (A^{-1})_{ik}. \tag{24}$$

Our aim is to show that

$$\langle \xi_i^\alpha F\rangle = \sum_k \left\langle \xi_i^\alpha\bar{\xi}_k^\beta\right\rangle\left\langle\frac{\partial F}{\partial\bar{\xi}_k^\beta}\right\rangle,$$
$$\langle \bar{\xi}_i^\alpha F\rangle = \sum_k \nu_{\alpha\beta}\left\langle\frac{\partial F}{\partial\xi_k^\beta}\right\rangle\left\langle \xi_k^\beta\bar{\xi}_i^\alpha\right\rangle. \tag{25}$$

Using (24), these can be rewritten as

$$\sum_k A_{ik}\langle \xi_k^\alpha F\rangle = \left\langle\frac{\partial F}{\partial\bar{\xi}_i^\beta}\right\rangle,$$
$$\sum_k \langle \bar{\xi}_k^\alpha F\rangle A_{ki} = \nu_{\alpha\beta}\left\langle\frac{\partial F}{\partial\xi_k^\beta}\right\rangle. \tag{26}$$

Consider the left-hand side of the first of the equations (26):

$$\begin{aligned}\sum_k A_{ik}\langle \xi_k^\alpha F\rangle &= \int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i A_{ik}\xi_k^\alpha F P(\xi,\bar{\xi})\\ &= \int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i \mathcal{P}(F) A_{ik}\xi_k^\alpha P(\xi,\bar{\xi})\\ &= -\int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i \mathcal{P}(F)\frac{\partial}{\partial\bar{\xi}_k^\beta}P(\xi,\bar{\xi})\\ &= \int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i\frac{\partial F}{\partial\bar{\xi}_k^\beta}P(\xi,\bar{\xi}) = \left\langle\frac{\partial F}{\partial\bar{\xi}_i^\beta}\right\rangle,\end{aligned} \tag{27}$$

where $\mathcal{P}(F)$ is obtained from F on changing the signs of the ξ_i^α's and $\bar{\xi}_i^\beta$'s in F. Now consider the left-hand side of the second equation in (26)

$$\begin{aligned}&\sum_k \langle \xi_i^\alpha F\rangle A_{ik}\\ &= \int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i\xi_i^\alpha F\exp\left(-\sum_{i,j}\sum_{\gamma,\delta}\bar{\xi}_i^\gamma A_{ij}\xi_j^\delta\right)A_{ik}\\ &= \int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i\mathcal{P}(F)\xi_i^\alpha\exp\left(-\sum_{i,j}\sum_{\gamma,\delta}\bar{\xi}_i^\gamma A_{ij}\xi_j^\delta\right)A_{ik}\\ &= -\int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i\mathcal{P}(F)\left(\frac{\partial}{\partial\xi_j^\delta}\exp\left(-\sum_{i,j}\sum_{\gamma,\delta}\bar{\xi}_i^\gamma A_{ij}\xi_j^\delta\right)\right)\\ &\quad\times\delta_{jk}\delta_{\gamma\alpha}\nu_{\gamma\delta}\\ &= \int\prod_i \mathrm{d}\bar{\xi}_i\mathrm{d}\xi_i\nu_{\alpha\delta}\frac{\partial F}{\partial\xi_k^\delta}\left(\exp\left(-\sum_{i,k}\sum_{\gamma,\delta}\bar{\xi}_i^\alpha A_{ik}\xi_k^\delta\right)\right)\\ &= \nu_{\alpha\beta}\left\langle\frac{\partial F}{\partial\xi_k^\beta}\right\rangle.\end{aligned} \tag{28}$$

Equations (27) and (28) are the results we set out to prove.

4 Equivalence between the Langevin and the Fokker–Planck formalisms

In order to establish connection between the Langevin and the Fokker–Planck formalisms, the averages in the two approaches are identified:

$$\begin{aligned}\left\langle F[\psi,\bar{\psi}]\right\rangle_{\eta\bar{\eta}} &= \int D\bar{\psi}D\psi F[\psi,\bar{\psi}]P[\psi,\bar{\psi},t]\\ &\equiv \left\langle F[\psi,\bar{\psi}]\right\rangle_P.\end{aligned} \tag{29}$$

It follows from (21), (27) and (28) that

$$\begin{aligned}\left\langle\frac{\partial F[\psi,\bar{\psi}]}{\partial t}\right\rangle_{\eta\bar{\eta}} &= \sum_\alpha\int \mathrm{d}^4x D\bar{\psi}(x)D\psi(x)F[\psi,\bar{\psi}]\\ &\times\left[\delta_{\alpha\gamma}\frac{\delta^2P[\psi,\bar{\psi},t]}{\delta\bar{\psi}^\gamma(x)\delta\psi^\alpha(x)} + \nu_{\gamma\alpha}\delta_{\alpha\gamma}\frac{\delta^2P[\psi,\bar{\psi},t]}{\delta\psi^\alpha(x)\delta\bar{\psi}^\gamma(x)}\right.\\ &+\sum_\beta\nu_{\alpha\beta}\frac{\delta}{\delta\psi^\alpha}\left(\frac{\delta S_{\mathrm{E}}[\psi,\bar{\psi}]}{\delta\bar{\psi}^\beta}P[\psi,\bar{\psi},t]\right)\\ &\left.+\sum_\beta\theta_{\alpha\beta}\frac{\delta}{\delta\bar{\psi}^\beta}\left(\frac{\delta S_{\mathrm{E}}[\psi,\bar{\psi}]}{\delta\psi^\alpha}P[\psi,\bar{\psi},t]\right)\right].\end{aligned} \tag{30}$$

Since F is an arbitrary function of ψ and $\bar{\psi}$, and using the fact that

$$\left\langle\frac{\partial F[\psi,\bar{\psi}]}{\partial t}\right\rangle_{\eta\bar{\eta}} = \int D\bar{\psi}D\psi F[\psi,\bar{\psi}]\frac{\partial P[\psi,\bar{\psi},t]}{\partial t}, \tag{31}$$

we obtain the Fokker–Planck equation (20).

We now apply the Langevin and Fokker–Planck equations constructed in (14) and (20) to free parafermi fields of order two satisfying the weak locality condition. The classical Euclidean action in the even sector is

$$S_{\mathrm{E}}^{(+)} = \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}t \times \left\{ -\sum_{\alpha} [\bar{\psi}^{\alpha}(x,t), (\mathrm{i}\nabla\!\!\!/ - m)\psi^{\alpha}(x,t)]_- + {\sum_{\alpha,\beta}}' \kappa[\bar{\psi}^{\alpha}(x,t), \psi^{\beta}(x,t)]_+ \right\}, \quad (32)$$

where the prime over the summation in the second term indicates that $\alpha \neq \beta$.

Performing a variation of the action with respect to the fields $\psi^{\alpha}(x,t)$ and $\bar{\psi}^{\alpha}(x,t)$ and substituting these back into the Langevin equations (14), we obtain

$$\frac{\partial\psi^{\alpha}(x,t)}{\partial t} = (\mathrm{i}\nabla\!\!\!/ - m - \kappa)\psi^{\alpha}(x,t) + \eta^{\alpha}(x,t),$$
$$\frac{\partial\bar{\psi}^{\alpha}(x,t)}{\partial t} = -\bar{\psi}^{\alpha}(x,t)(\mathrm{i}\overleftarrow{\nabla\!\!\!/} + m + \kappa) + \bar{\eta}^{\alpha}(x,t). \quad (33)$$

For the initial conditions

$$\psi^{\alpha}(k,0) = \bar{\psi}^{\alpha}(k,0) = 0, \quad (34)$$

their solution in momentum space is given by

$$\psi^{\alpha}(k,t) = \int \mathrm{d}t_1 \mathrm{e}^{-(k\!\!\!/+m+\kappa)(t-t_1)}\theta(t-t_1)\eta^{\alpha}(k,t_1),$$
$$\bar{\psi}^{\alpha}(k',t') = \int \mathrm{d}t_1 \bar{\eta}^{\alpha}(k',t_1)\mathrm{e}^{-(-k\!\!\!/'+m+\kappa)(t'-t_1)}\theta(t'-t_1). \quad (35)$$

From this one finds that the stochastic propagator for the order two parafermi field in the even sector is, after performing a summation over all the Green components in the end,

$$\left\langle \psi(k,t)\bar{\psi}(k',t')\right\rangle = \sum_{\alpha,\beta}\left\langle \psi^{\alpha}(k,t)\bar{\psi}^{\beta}(k',t')\right\rangle \quad (36)$$
$$= \sum_{\alpha,\beta}\delta_{\alpha\beta}\frac{\delta(k+k')}{(k\!\!\!/ + m + \kappa)} \times \left(\mathrm{e}^{-(k\!\!\!/+m+\kappa)|t-t'|} - \mathrm{e}^{-(k\!\!\!/+m+\kappa)(t+t')}\right).$$

Setting $t' = t$ we obtain in the equilibrium limit $t \to \infty$ the quantum propagator

$$\lim_{t\to\infty}\sum_{\alpha,\beta}\left\langle \psi^{\alpha}(k,t)\bar{\psi}^{\beta}(k',t)\right\rangle = \sum_{\alpha=1}^{2}\frac{\delta(k+k')}{(k\!\!\!/ + m + \kappa)} = \frac{2\delta(k+k')}{(k\!\!\!/ + m + \kappa)}. \quad (37)$$

Similarly one obtains for the quantum propagator for an order two parafermi field in the odd sector, the result

$$\lim_{t\to\infty}\left\langle \psi(k,t)\bar{\psi}(k',t)\right\rangle = \lim_{t\to\infty}\sum_{\alpha,\beta}\left\langle \psi^{\alpha}(k,t)\bar{\psi}^{\beta}(k',t)\right\rangle = \frac{2\delta(k+k')}{(k\!\!\!/ + m - \kappa)}. \quad (38)$$

Thus we see that a parafermi field of order two gives rise to two fields of masses $(m+\kappa)$ and $(m-\kappa)$, a result which is in agreement with that obtained from the canonical quantization procedure. It is easy to confirm that the same results for the propagators are obtained with the Fokker–Planck equation (20).

5 Equivalence of the modified stochastic equations to Euclidean field theory in the equilibrium limit

In order to prove that the stochastic averages obtained from the modified Langevin equations (14) and the modified Fokker–Planck equation (20) relax to the quantum averages in the steady state limit, one has to show that the Fokker–Planck Hamiltonian H_{FP} is indeed positive definite. To prove this for a parafermi field of order two, it suffices to use the known result that such a field in fact describes two ordinary Fermi fields of masses $(m+\kappa)$ and $(m-\kappa)$ through Klein transformations. This was shown by Ohnuki and Kamefuchi [3] by introducing a non-local Klein operator K_2^1, which enables the parafermi field $\psi(x)$ to be written in terms of two ordinary Fermi fields $\phi^{\alpha}(x)$, $(\alpha = 1,2)$:

$$\psi(x) = \phi^{(1)}(x) - \mathrm{i}K_2^1\phi^{(2)}(x),$$
$$\psi^{\dagger}(x) = {\phi^{(1)}}^{\dagger}(x) - \mathrm{i}K_2^1{\phi^{(2)}}^{\dagger}(x), \quad (39)$$

where $K_2^1 \equiv (-1)^N$, N being the number operator. Letting

$$\Phi^{(1)}(x) \equiv \frac{1}{\sqrt{2}}(\phi^{(1)}(x) + \mathrm{i}\phi^{(2)}(x)),$$
$$\Phi^{(2)}(x) \equiv \frac{1}{\sqrt{2}}(\phi^{(1)}(x) - \mathrm{i}\phi^{(2)}(x)), \quad (40)$$

the total action for the parafermi field may be thought of as the sum of two different actions $S_{(1)}$ and $S_{(2)}$ describing the fields $\Phi^{(1)}$ and $\Phi^{(2)}$ respectively:

$$S_{\mathrm{E}} = S_{(1)} + S_{(2)},$$
$$S_{(1)} = -\frac{1}{2}\int \mathrm{d}^4x[\bar{\Phi}^{(1)}, (\mathrm{i}\nabla\!\!\!/ - m - \kappa)\Phi^{(1)}]_- = \int \mathrm{d}^4x\bar{\Phi}^{(1)}K_1\Phi^{(1)},$$
$$S_{(2)} = -\frac{1}{2}\int \mathrm{d}^4x[\bar{\Phi}^{(2)}, (\mathrm{i}\nabla\!\!\!/ - m + \kappa)\Phi^{(2)}]_- = \int \mathrm{d}^4x\bar{\Phi}^{(2)}K_2\Phi^{(2)}, \quad (41)$$

with

$$K_1 = -(\mathrm{i}\nabla\!\!\!/ - m - \kappa),$$
$$K_2 = -(\mathrm{i}\nabla\!\!\!/ - m + \kappa). \quad (42)$$

The fields $\Phi^{(i)}$, $\bar{\Phi}^{(i)}$, $(i = 1, 2)$ satisfy the ordinary bilinear commutation rules

$$[\Phi_m^{(i)}, \bar{\Phi}_n^{(i)\dagger}]_+ = \delta_{mn},$$
$$[\Phi_m^{(i)}, \bar{\Phi}_n^{(i)}]_+ = 0, \quad \text{etc.}, \tag{43}$$

where m and n are spinor indices.

Keeping in mind the fact that for an order two parafermi field, the two fields $\Phi^{(1)}$ and $\Phi^{(2)}$ are mutually inclusive, i.e, the presence of one of the $\Phi^{(i)}$ fields necessarily implies the presence of the other one also, the total probability $P[\Phi^{(1)}, \Phi^{(2)}, \bar{\Phi}^{(1)}, \bar{\Phi}^{(2)}, t]$ of finding the parafermi system in the given configuration can be written as

$$P[\Phi^{(1)}, \Phi^{(2)}, \bar{\Phi}^{(1)}, \bar{\Phi}^{(2)}, t] = P[\Phi^{(1)}, \bar{\Phi}^{(1)}, t] + P[\Phi^{(2)}, \bar{\Phi}^{(2)}, t]. \tag{44}$$

The Fokker–Planck equation for the parafermi field can then be written as

$$\frac{\partial P}{\partial t} = -H_{\mathrm{FP}_1} P_1 - H_{\mathrm{FP}_2} P_2 = -H_{\mathrm{FP}} P, \tag{45}$$

where

$$H_{\mathrm{FP}_i} = -\left[\frac{\delta}{\delta\bar{\Phi}^{(i)}}\left(\frac{\delta}{\delta\Phi^{(i)}} + \frac{\delta S}{\delta\Phi^{(i)}}\right) - \frac{\delta}{\delta\Phi^{(i)}}\left(\frac{\delta}{\delta\bar{\Phi}^{(i)}} + \frac{\delta S}{\delta\bar{\Phi}^{(i)}}\right)\right] \quad (i = 1, 2), \tag{46}$$

which on applying (41) becomes

$$H_{FP_i} = -\left[\frac{\delta}{\delta\bar{\Phi}^{(i)}}\left(\frac{\delta}{\delta\Phi^{(i)}} - {K_i}^T\bar{\Phi}^{(i)}\right) - \frac{\delta}{\delta\Phi^{(i)}}\left(\frac{\delta}{\delta\bar{\Phi}^{(i)}} + K_1\Phi^{(i)}\right)\right] \quad (i = 1, 2). \tag{47}$$

The grassmann fields $\Phi^{(k)}$, $\bar{\Phi}^{(k)}$ are formally defined in terms of an infinite number of grassmann numbers ξ_i, $\bar{\xi}_i$ as follows:

$$\Phi^{(k)}(x) = \sum_{i=1}^{\infty} f_i(x)\xi_i, \bar{\Phi}^{(k)}(x) = \sum_{i=1}^{\infty} f_i^*(x)\bar{\xi}_i \quad (k = 1, 2). \tag{48}$$

$f_i(x)$ are a complete set of orthonormal functions. The fermion actions $S_{(1)}$, $S_{(2)}$ can be written in terms of these independent paragrassmann variables ξ_i, $\bar{\xi}_i$ as

$$S_{(1,2)} = \sum_{i,j} \bar{\xi}_i K_{1,2_{ij}} \xi_j, \tag{49}$$

where

$$K_{1,2_{ij}} = \int \mathrm{d}^4 x f_i^*(x) K_{1,2} f_j(x). \tag{50}$$

In view of the definition (48) for Grassmann fields, the operators $\delta/\delta\bar{\Phi}^{(i)}$ and $\delta/\delta\Phi^{(i)}$ are defined by

$$\frac{\delta}{\delta\Phi^{(k)}} = \sum_i f_i^*(x)\frac{\partial}{\partial\xi_i},$$
$$\frac{\delta}{\delta\bar{\Phi}^{(k)}} = \sum_i f_i(x)\frac{\partial}{\partial\bar{\xi}_i} \quad (k = 1, 2), \tag{51}$$

so that the Fokker–Planck hamiltonian (47) takes the form

$$H_{\mathrm{FP}_k} = -\sum_i \left[\frac{\delta}{\delta\bar{\xi}_i}\left(\frac{\delta}{\delta\xi_i} - \sum_l K_{k_{li}}\bar{\xi}_l\right) - \frac{\delta}{\delta\xi_i}\left(\frac{\delta}{\delta\bar{\xi}_i} + \sum_l K_{k_{il}}\xi_l\right)\right] \quad (k = 1, 2). \tag{52}$$

Following the work of Fukai et al. [5], we introduce the following abstract representations:

$$\xi_i, \bar{\xi}_i \to A_i, B_i,$$
$$\frac{\partial}{\partial\xi_i}, \frac{\partial}{\partial\bar{\xi}_i} \to A_i^\dagger, B_i^\dagger, \tag{53}$$

and the coherent states of Fermi operators:

$$|\xi, \bar{\xi}\rangle = \prod_{j=1}^{\infty}(1 - \xi_j A_j^\dagger) \prod_{j=1}^{\infty}(1 - \bar{\xi}_j B_j^\dagger)|0\rangle. \tag{54}$$

These coherent states form a complete orthonormal set [11]. Here $A_i^\dagger$ and $B_i^\dagger$ are the normal Fermi creation operators satisfying the algebra

$$[A_i, A_i^\dagger]_+ = [B_i, B_i^\dagger]_+ = 1,$$
$$[A_i, A_j]_+ = [B_i, B_j]_+ = 0, \tag{55}$$

and the destruction operators A_i and B_i annihilate the vacuum $|0\rangle$:

$$A_i|0\rangle = B_i|0\rangle = 0. \tag{56}$$

Having simplified the original parafield system considerably by writing it in terms of ordinary Fermi fields by the above procedure, we can now show that the Fokker–Planck hamiltonian (45), (46) is indeed positive definite. Equation (45) can be written in the form of an eigenvalue equation:

$$H_{\mathrm{FP}}\chi_n = \sum_n \lambda_n \chi_n. \tag{57}$$

In terms of the eigenfunctions χ_n and the corresponding eigenvalues of the operator H_{FP}, the general solution of (44) is given by

$$P = \sum_n \chi_n \mathrm{e}^{-\lambda_n t}. \tag{58}$$

In order to examine the spectrum of H_{FP}, we regard the probability distribution $P_k(\xi, \bar{\xi}, t)$ as the representative of an abstract vector $|P_k(t)\rangle$ in the coherent state representation, so that the Fokker–Planck equation for the field $\Phi^{(k)}$ assumes the form

$$\frac{\partial}{\partial t}|P_k(t)\rangle = -H_{\mathrm{FP}_k}|P_k(t)\rangle \quad (k = 1, 2), \tag{59}$$

where

$$H_{\mathrm{FP}_k} = -\sum_i \left[B_i^\dagger \left(A_i^\dagger - \sum_l K_{k_{li}} B_l \right) - A_i^\dagger \left(B_i^\dagger - \sum_l K_{k_{il}} A_l \right) \right]. \quad (60)$$

For diagonalizing the eigenvalue equation corresponding to (59), we perform the similarity transformation

$$\hat{H}_{\mathrm{FP}_k} = \exp \sum_{ij} (A_i^\dagger {K_{k_{il}}}^{-1} B_l^\dagger) H_{\mathrm{FP}_k} \times \exp \left(- \sum_{ij} A_i^\dagger {K_{k_{il}}}^{-1} B_l^\dagger \right), \quad (61)$$

under which

$$A_i^\dagger \to A_i^\dagger, \quad B_i^\dagger \to B_i^\dagger,$$
$$A_i \to A_i - {K_{k_{il}}}^{-1} B_l^\dagger, \quad B_i \to B_i - {K_{k_{li}}}^{-1} A_l^\dagger. \quad (62)$$

We find that the operator $\hat{H}_{\mathrm{FP}_k}$ is given by

$$\hat{H}_{\mathrm{FP}_k} = \sum_i [B_i^\dagger (K_k^T)_{il} B_l + A_i^\dagger K_{k_{il}} A_l]. \quad (63)$$

It is evident from this that as long as K_k is a positive definite operator, $\hat{H}_{\mathrm{FP}_k}$, and hence also H_{FP_k} are both positive definite. This in turn implies that H_{FP} for the entire parafermi system is positive definite. Therefore, when the limit $t \to \infty$ is taken, every solution $P(\xi, \bar{\xi}, t)$ in (58) of the Fokker–Planck equation (57) approaches the eigenfunction of $\hat{H}_{\mathrm{FP}}$ corresponding to the zero eigenvalue provided this eigenvalue is non-degenerate and there is a gap in the spectrum of H_{FP} above the zero eigenvalue.

We have thus shown that the stochastic quantization method with the modified Langevin and the Fokker–Planck equations (14) and (20) respectively, for an order two parafermi field leads to the usual quantum field theory in the equilibrium limit $t \to \infty$. Although a parafermi field of order two has been discussed here, it must be pointed out that the modified Langevin equation (14) as well as the modified Fokker–Planck equation (20) can both be applied to parafermi fields of all orders. For a parafermi field of odd order, the off-diagonal terms on the right-hand sides of (14) and (20), that is, terms corresponding to Green index $\alpha \neq \beta$ will not contribute, there being only commutator terms in the action for such fields.

6 Discussion

We have shown in this work how to extend the stochastic quantization method of Parisi–Wu to include in four space-time dimensions, parafermi fields of order two satisfying the weak locality condition, by modifying the Langevin and the corresponding Fokker–Planck equations in an appropriate manner. It is shown that in the equilibrium limit $t \to \infty$, the usual quantum field theory for an order two parafermi field is recovered, giving rise to two fields of masses $(m+\kappa)$ and $(m-\kappa)$. As mentioned towards the end of the previous section, the modified Langevin and Fokker–Planck equations constructed here can be applied also to parafermi fields of all orders. As far as we know, stochastic quantization of parabose fields has not been done yet – we have confined this work to the case of order two parafermi fields for the sake of completeness while discussing parafermi fields of all orders within the Parisi–Wu framework – it is interesting to know how to modify the stochastic equations to handle both commutator and anti-commutator terms in the classical action simultaneously.

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